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DERIVATION OF THE BOUNDARY CONDITIONS FOR THE EQUATION OF THE VIBRATIONS OF A THIN PLATE BY THE METHOD OF GENERALIZED FUNCTIONS†

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The boundary conditions on the free boundary of a thin vibrating plate of variable thickness, when the thickness of the plate is changed abruptly, are derived. The left-hand side of the fourth-order differential equation describing the vibrations of the plate has a singularity of the δ-function type and its derivative. Since the right-hand side of this equation has no singularity, it is natural to equate the coefficients of the generalized functions to zero. These equations also represent the boundary conditions. This procedure for finding the boundary conditions is not new [1, 2], but its application to a fourth-order equation involves the need to take partial derivatives of the δ-function distributed over the contour. A formula for calculating such derivatives is derived and is used to obtain the boundary conditions.

THE TRADITIONAL method of solving problems of this type [3, 4] consists of using the variational principle. The integral over the surface of the plate is transformed into an integral over its edge, which is extremely time consuming. The approach proposed below is simpler and physically clearer.

1. FORMULATION OF THE PROBLEM

We will start from the equation of motion of thin plates (see, for example, [5])

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}=-m(x,y)\omega^2 w$$

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\quad M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right) \tag{1.1}$$

$$M_{xy}=D(1-\nu)\frac{\partial^2 w}{\partial x\partial y};\quad D=\frac{Eh^3}{12(1-\nu^2)},\quad m=\mu h$$

Here w is the transverse displacement of the plate, h is its thickness, μ is the density, ω is the angular frequency of the vibrations, and E and ν are the elastic constants.

Note that Eq. (1.1) holds for an arbitrary distribution of the density, thickness and elastic constants along the surface of the plate. In particular, it remains true when the thickness of the plate falls to zero on

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passing through the edge. The moments M_x, M_y and M_{xy} are then discontinuous functions. Consequently, their second derivatives will contain a δ-function and its derivative. Since the right-hand side of Eq. (1.1) does not contain a singularity, the coefficients of the generalized functions must be equated to zero. Hence, the problem reduces to calculating the derivatives of the moments under conditions when the thickness h has a discontinuity.

2. DERIVATION OF THE BOUNDARY CONDITIONS

The moments can obviously be written in the form

$$M(x,y)=\begin{cases} M^*(x,y) & \text{inside the edge} \\ 0 & \text{outside the edge} \end{cases} \tag{2.1}$$

Hence and henceforth the asterisk will denote values of quantities on the plate, i.e. the "usual" moments and their "usual" derivatives (the limits of these derivatives when the point approaches the edge).

Suppose $\mathbf{n}$ is the inward unit normal to the edge and $\mathbf{l}$ is the unit tangential vector in the positive direction of circumventing the plate.

The first partial derivatives of functions of the type (2.1) are given by the formulae [6]

$$\partial M/\partial z=(\partial M/\partial z)^*+\delta(\Gamma)\cos(\mathbf{n},\mathbf{z}),\quad z=x,y \tag{2.2}$$

where Γ is the edge of the plate and $\delta(\Gamma)$ is the distributed δ-function.

On repeated differentiation of one of the expressions in (2.2) the first term, representing a discontinuous function, allows the use of equations of the form (2.2). The calculation of the derivative of the second term requires a special approach (see the Appendix)

$$\frac{\partial}{\partial z}[\rho(\Gamma)\delta(\Gamma)]=-\rho(\Gamma)\delta'(\Gamma)\cos(n,z)+\frac{\partial}{\partial l}[\delta(\Gamma)\cos(l,z)]\delta(\Gamma) \tag{2.3}$$

In place of $\rho(\Gamma)$ we can put $M^*(\Gamma)$. The symbol $\delta'(\Gamma)$ denotes a function of the double-layer type. The determination and properties of the functions δ and δ' are discussed in the Appendix.

Using (2.2) and (2.3) we can calculate the second derivatives in Eq. (1.1). We will introduce the angle φ between $\mathbf{x}$ and $\mathbf{n}$ in a positive direction. Equating the coefficients of δ' and δ to zero in Eq. (1.1), we obtain the required boundary conditions

$$-[M_x\cos^2\varphi+M_y\sin^2\varphi-2M_{xy}\sin\varphi\cos\varphi]=0 \tag{2.4}$$

$$\begin{gathered}\frac{\partial M_x}{\partial x}\cos\varphi+\frac{\partial M_y}{\partial y}\sin\varphi-\left(\frac{\partial M_{xy}}{\partial x}\sin\varphi+\frac{\partial M_{xy}}{\partial y}\cos\varphi\right)+ \\ +\frac{\partial}{\partial l}[\cos\varphi\sin\varphi(M_x-M_y)-M_{xy}(\sin^2\varphi-\cos^2\varphi)]=0\end{gathered} \tag{2.5}$$

The asterisk over the moments and their derivatives is omitted everywhere.

3. DISCUSSION OF THE RESULTS

We will compare the boundary conditions (2.4) and (2.5) with the known boundary conditions. We first note that the boundary conditions obtained agree with those derived in [4]. For the special case of a rectilinear edge, situated along the axis (see, for example, [5, 7])

$$M_x = 0, \quad \partial M_x / \partial x - 2\partial M_{xy} / \partial y = 0 \tag{3.1}$$

These expressions follow from (2.4) and (2.5) when $\varphi = 0$.

The boundary conditions for an edge of arbitrary shape for a constant plate thickness [3] follow from (2.4) and (2.5) if we put $h = \text{const}$ and assume $\mathbf{n}$ to be the outward normal.

Note that the method proposed enables one to obtain the boundary conditions on the free edge for any equation of the theory of elasticity. It is merely necessary that this equation should be used in the case of variable stiffness.

On the other hand, the boundary conditions written in the form (2.4), (2.5) enable the methods of potential theory to be employed to solve boundary-value problems of the theory of thin plates. Suppose we know the Green's function of Eq. (1.1), i.e. the function $w = G(P_1, P_2)$, such that

$$L[w] \equiv \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \omega^2 m w = \delta(P_1 - P_2)$$

(P_1 and P_2 are points on the plate). Differentiation is carried out with respect to the coordinates of the point P_1. In a number of cases, for example, for an acute-angled elastic wedge, it is possible to obtain the function G.

Suppose that we are given boundary conditions of the following form on the edge of the plate

$$B_k[w] = \psi_k(\Gamma), \quad k = 1, 2$$

where the operators B_1 and B_2 are the left-hand sides of Eqs (2.4) and (2.5), respectively. The solution of this problem will be sought in the form of a linear combination of potentials of a single and double layer

$$w(P_1) = \int_\Gamma f_1(P_2) \frac{\partial}{\partial n} G(P_1, P_2) d\Gamma + \int_\Gamma f_2(P_2) G(P_1, P_2) d\Gamma, \quad P_2 \in \Gamma$$

where $\mathbf{n}$ is the inward normal to the edge with respect to the coordinates of the point P_2. The integration is carried out over the coordinates of the point P_2. It can be seen that

$$L[w] = f_1(\Gamma)\delta'(\Gamma) + f_2(\Gamma)\delta(\Gamma)$$

According to the above discussion, the functions f_1 and f_2 define discontinuities in the values of the operators B_1 and B_2 on the edge of the plate. From general considerations, similar to those used in potential theory, on the edge itself the operators B_1 and B_2 give the arithmetic mean values on the right and on the left. Hence, the boundary-value problem reduces to a system of Fredholm equations of the second kind

$$\frac{1}{2} f_k(P_1) + \int_\Gamma B_k [f_1(P_2) \partial G(P_1, P_2) / \partial n + f_2(P_2) G(P_1, P_2)] d\Gamma = \psi_k(P_1)$$

$$k = 1, 2, \quad P_1, P_2 \in \Gamma$$

APPENDIX

The distributed generalized functions δ and δ' are specified as follows. We will assume that the region S, the contour Γ inside it, and an arbitrary smooth function Φ, equal to zero outside S, are given in the plane. Then

$$\int_S \Phi\delta(\Gamma)dS = \int_\Gamma \Phi(\Gamma)d\Gamma, \quad \int_S \Phi\delta'(\Gamma)dS = \int_\Gamma \frac{\partial\Phi}{\partial n}(\Gamma)d\Gamma \tag{A.1}$$

Suppose $\rho(\Gamma)$ is a function given on the contour Γ and having the meaning of the weight of the δ-function. We will calculate the partial derivative of $\rho\delta(\Gamma)$ with respect to $z(z = x,\ y)$. Integrating by parts, we obtain

$$\begin{aligned}\int_S \Phi\frac{\partial}{\partial z}[\rho\delta(\Gamma)]dS &= -\int_S \rho\delta(\Gamma)\frac{\partial\Phi}{\partial z}dS = -\int_\Gamma \rho\frac{\partial\Phi}{\partial z}(\Gamma)d\Gamma = \\ &= -\int_\Gamma \rho(\Gamma)\cos(n,z)\frac{\partial\Phi}{\partial n}d\Gamma + \int_\Gamma \Phi\frac{\partial}{\partial l}[\rho(\Gamma)\cos(l,z)]d\Gamma\end{aligned} \tag{A.2}$$

Comparing this result with (A.1), by virtue of the arbitrary nature of Φ we can conclude that (2.3) holds.

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